For the modified theory, the temperature-wave velocity is again described by Eqs. (20) and (21), but in this case the definitions of a, c, and m are as follows:

$$a = -\delta_G \hat{g}(\theta, \bar{\theta}^t, \bar{G}^t|n\ 1^+), \tag{25}$$

$$c = D_\theta \hat{e}(\theta, \bar{\theta}^t, \bar{G}^t), \tag{26}$$

$$m = \frac{1}{U_0 c} \delta_g \hat{e}(\theta, \bar{\theta}^t, \bar{G}^t|n\ 1^+). \tag{27}$$

In [1], Eqs. (20)-(24) formed the basis for the conclusion that if

$$\delta_g e(\vartheta, \bar{\vartheta}^t, \bar{g}|n\ 1^+) \approx 0 \tag{28}$$

for all n, the temperature-wave velocity in the direction of **q** was larger than in the direction **−q**. Thus, this velocity is not simply a property of the material but is a function of the process. As follows from Eqs. (25)-(27), this effect is absent from the modified theory and, when only Eq. (28) is satisfied (if, for example, the material has a center of symmetry), $U = U_0$. The velocity U_0 may be regarded in the normal sense as a characteristic of the material since calculations of U_0 retaining only the main linear terms give a constant which depends solely on the temperature.

It should be emphasized that the difference of principle between the two theories are confirmed by experimental verification.

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HEAT TRANSFER IN SEMIINFINITE REGION WITH VARIABLE PHYSICAL PARAMETERS

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A method is proposed for the determination of the nonsteady temperature field in a semiinfinite region with variable physical properties.

The heating of a semiinfinite region with variable physical parameters in the coordinate and the time, for zero initial conditions, may be described by the following equation

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \gamma(x, t)\right] T = 0, \quad 0 \leqslant x < \infty, \quad 0 < t < \infty, \tag{1}$$

$$T|_{x=0} = T_0(t); \quad T|_{x=\infty} = 0; \quad T|_{t=0} = 0.$$

It is required to find the temperature field T(x, t).

Earlier, for an analogous problem, only the temperature gradient at the boundary $(\partial T/\partial x)_{x=0}$ was found [1, 2].

The total solution of Eq. (1) will be sought in the form of a functional series

$$T = \sum_{n=0}^{\infty} c_n(x, t) D^{-n/2} e^{-xD^{1/2}} T_0(t). \tag{2}$$

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Here D^ν are fractional-derivative operators. For an arbitrary function f(t) [3]

$$D^\nu f(t) = \frac{1}{\Gamma(1-\nu)} \cdot \frac{d}{dt}\int_0^t (t-\tau)^{-\nu} f(\tau)\,d\tau, \quad -\infty < \nu < 1,$$
$$D^\nu D^\mu f(t) = D^{\nu+\mu} f(t), \quad \nu+\mu \leqslant 1, \tag{3}$$
$$D^\nu t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}\, t^{\mu-\nu}.$$

The operator $e^{-xD^{1/2}}$ is given by the expression

$$e^{-xD^{1/2}} f(t) = \frac{d}{dt}\int_0^t \left[1-\Phi\left(\frac{x}{2\sqrt{t-\tau}}\right)\right] f(\tau)\,d\tau. \tag{4}$$

The rather unwieldy form in which the operator is given emphasizes its properties

$$\frac{d}{dx} e^{-xD^{1/2}} f(t) = -D^{1/2} e^{-xD^{1/2}} f(t) = -e^{-xD^{1/2}} D^{1/2} f(t),$$
$$D^\nu e^{-xD^{1/2}} f(t) = e^{-xD^{1/2}} D^\nu f(t), \tag{5}$$
$$D^{-1/2} e^{-xD^{1/2}} f(t) = \int_x^\infty [e^{-xD^{1/2}} f(t)]\,dx,$$

which are verified by means of a Laplace transform with respect to t. The function f(t) is assumed to be bounded and piecewise smooth: $\lim_{t\to+0} f(t) \leqslant \text{const}$.

Substituting Eq. (2) into Eq. (1), using Eqs. (4) and (5), and equating terms of the same order in $D^{-n/2}\cdot e^{-xD^{1/2}}\,T_0(t)$ gives a system of recurrence relations for c_n

$$c_0 = 1; \quad c_1 = -\frac{1}{2}\int_0^x \gamma(x, t)\,dx;$$
$$c_{n+1} = -\frac{1}{2}\int_0^x \left(\frac{\partial c_n}{\partial t} + \gamma c_n - \frac{\partial^2 c_n}{\partial x^2}\right) dx. \tag{6}$$

Equation (2) satisfies all the conditions in Eq. (1) if the series converges and may be differentiated term by term. In fact, Eq. (2) gives $T = T_0(t)$ for $x = 0$ and $T = 0$ for $x = \infty$. It is also known that the function $T = e^{-xD^{1/2}}T_0(t)$, being a solution of Eq. (1) for $\gamma = 0$, satisfies the zero initial condition for $0 < x < \infty$. For analytic $\gamma(x, t)$, the use in this solution of the operator of Eq. (2) does not change the order of $\lim_{t\to+0} T$; i.e., Eq. (2) satisfies the zero initial conditions.

Thus, the series in Eq. (2) is the solution of Eq. (1). If $\gamma(x)$ depends only on x, the solution may be obtained by means of a Laplace transform, since $D^\nu f(t) = p^\nu \bar{f}(p)$; $e^{-xD^{1/2}}f(t) = e^{-x\sqrt{p}}\bar{f}(p)$.

Example 1. Let $\gamma = 2(1+x)^{-2}$; $T_0(t)$ is an arbitrary function. Then Eq. (6) gives $c_0 = 1$; $c_n = (-1)^n x(1+x)^{-1}$, $n \geq 1$. Summing the Laplace transform of Eq. (2) gives

$$\bar{T} = \left(1 - \frac{1}{1+\sqrt{p}} \cdot \frac{x}{1+x}\right) e^{-x\sqrt{p}}\, \bar{T}_0(p).$$

Direct verification confirms this solution.

The confidence that the method is appropriate for any function of the arguments $\gamma(x, t)$ that is everywhere analytic derives from the verification of a number of complex examples, one of which is given below.

Example 2. Let $\gamma = x^2/4(t^2-1) + t/2$. In this case, an accurate solution of Eq. (1) is known

$$T = e^{x^2 t/4}\left[1 - \Phi\left(x/2\sqrt{e^{-t^2}\int_0^t e^{t^2}\,dt}\right)\right]. \tag{7}$$

It will be instructive to compare this with the solution obtained by the present method. From Eq. (6)

$$c_1 = -\frac{x^4}{24}(t^2-1) - \frac{xt}{4};$$
$$c_2 = \frac{x^6}{1152}(t^4-t^2+1) + \frac{x^4t^3}{96} - \frac{x^2}{32}(t^2-4);$$
$$c_3 = -\frac{x^9}{82944}(t^6-2t^4+2t^2-1) - \frac{x^7}{32256}(7t^5+t^3-12t) + \tag{8}$$
$$+\frac{x^5}{3840}(11t^4-37t^2+22) + \frac{9}{384}x^3t^3 - \frac{x}{32}(t^2-4).$$

Since Eqs. (2) and (7) are of completely different form, the comparison made will be $(\partial T/\partial x)_{x=0}$. From Eq. (7)

$$-\left.\frac{\partial T}{\partial x}\right|_{x=0} = \left(\pi e^{-t^2}\int_0^t e^{t^2}\,dt\right)^{-1/2} = \pi^{-1/2}\left(t^{-1/2} + \frac{t^{3/2}}{3} + \frac{t^{7/2}}{30} + \cdots\right). \tag{9}$$

Using Eqs. (5) and (8) and calculating the necessary component terms $c_4, \ldots, c_7$, Eq. (2) gives

$$-\left.\frac{\partial T}{\partial x}\right|_{x=0} = D^{1/2}\,1 - \sum_{n=1}^{\infty}\left.\frac{\partial c_n}{\partial x}\right|_{x=0} D^{-n/2}\,1 =$$
$$= \frac{t^{-1/2}}{\Gamma(1/2)} + \frac{t}{4}\cdot\frac{t^{1/2}}{\Gamma(3/2)} + \frac{t^2-4}{32}\cdot\frac{t^{3/2}}{\Gamma(5/2)} - \frac{5t^3}{128}\cdot\frac{t^{5/2}}{\Gamma(7/2)} -$$
$$-\left(\frac{21t^4}{2048} - \frac{11t^2}{64} + \frac{21}{384}\right)\frac{t^{7/2}}{\Gamma(9/2)} + O(t^{9/2}) + \cdots.$$

After grouping terms in increasing powers of t, this expression accurately gives the first three terms of Eq. (9).

NOTATION

T	is the temperature;
T_0	is the temperature in boundary region;
x	is the coordinate;
t	is the time;
γ	is the heat-transfer function;
c_n	are the unknown functions;
f	is the arbitrary function;
D^ν	is the fractional-differentiation operator;
p	is the Laplacian;
Φ	is the probability integral;
ν, μ, n	are the differentiation and summation indices.

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